



# Discrete Mathematics (011122)



魏可佶

kejiwei@tongji.edu.cn



1.1 List of Symbols

1.2 Sets and Operations

1.3 Overview of Proof Methods

1.4 Recursive Definitions

- Set definition and its notation
- Subset and equality of sets
- Empty set and universal set
- Set operations (Intersection  $\cap$ , Union  $\cup$ , Relative complement  $-$ ,
- Symmetric difference  $\oplus$ , Absolute complement  $\sim$ )
- Venn diagram representation of set operations
- Basic identities of set operations
- Proof of partial identities

## Set Definition

### ■ Cantor and Russell's discussions:

- A set is one of the most fundamental concepts in mathematics, with no rigid definition. It is understood as a **collection of elements**, often denoted by  $A$ ,  $B$ , etc.

### ■ Element: The individual components of a set.

- $x \in A$  ( $x$  is an element of  $A$ )
- $x \notin A$  ( $x$  is not an element of  $A$ )

### ■ Finite Set: A set with a limited number of elements.

### ■ Infinite Set: A set with an unlimited number of elements.

### ■ Cardinality of Set $|A|$ : The number of elements in set $A$ .

### ■ $k$ -element Set: A set containing $k$ elements, where $k \geq 0$ .

- **List Notation (enumeration method):**

For example,  $A=\{a,b,c,d\}$ ,  $N=\{0,1,2,\dots\}$

- **Set Builder Notation (descriptive method):**

For example,  $\{x \mid P(x)\}$ , where  $x$  is a natural number.

- **Explanation:**

(1) The elements in the set are **distinct**. For example,  $\{1,2,3\}=\{1,1,2,3\}$

(2) The elements in the set **do not have an order**. For example

$$\{1,2,3\}=\{3,1,2\}=\{1,3,1,2,2\}$$

(3) The two methods should be selected based on the context.

- **Common Sets:**

Natural numbers **N** (including 0), integers **Z**, positive integers **Z<sup>+</sup>**, rational numbers **Q**, irrational numbers **Q'**, real numbers **R**, all non-zero real numbers sets **R<sup>\*</sup>**, complex numbers **C**, intervals **[a,b]**, **(a,b)** etc.

■ **Subset (Inclusion):**  $A \subseteq B \Leftrightarrow \forall x (x \in A \rightarrow x \in B)$

■ **Not a Subset:**  $A \not\subseteq B \Leftrightarrow \exists x (x \in A \wedge x \notin B)$

■ **Equality of Sets:**  $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$

■ **Not Equal:**  $A \neq B \Leftrightarrow A \not\subseteq B \vee B \not\subseteq A$

■ **Proper Subset (Strict Subset):**  $A \subset B \Leftrightarrow A \subseteq B \wedge A \neq B$

Example:  $A = \{1, 2, 3\}$ ,  $B = \{x \mid x \in \mathbb{R} \wedge |x| \leq 1\}$ ,  $C = \{x \mid x \in \mathbb{R} \wedge x^2 = 1\}$ ,  $D = \{-1, 1\}$ , then we have  $C \subseteq B$ ,  $C \subset B$ ,  $C \not\subseteq A$ ,  $A \not\subseteq B$ ,  $B \not\subseteq A$ ,  $C = D$

■ **Properties:**

(1)  $A \subseteq A$

(2)  $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$

- **Empty set  $\emptyset$** : A set that contains no elements.

**Example:**  $\{x \mid x^2 < 0 \wedge x \in \mathbf{R}\} = \emptyset$

- **Theorem 1.1:** The empty set is a subset of any set.

**Proof:** By contradiction.

Suppose the contrary, i.e., there exists a set **A**, such that  $\emptyset \not\subseteq A$ , This means there exists an element **x**,  $x \in \emptyset$  and  $x \notin A$ , which is a contradiction.

- **Corollary:** The empty set is unique.

**Proof:** Suppose there exist two empty sets  $\emptyset_1, \emptyset_2$ , then  $\emptyset_1 \subseteq \emptyset_2$  and  $\emptyset_2 \subseteq \emptyset_1$ , Hence  $\emptyset_1 = \emptyset_2$

- **Universal set E:** A set *E* is called the universal set if all sets discussed in a problem is subset of *E*, then  $\forall A (A \subseteq E)$ , then *E* is referred to as the universal set.

- **(Power Set)** It refers to the set of all subsets of a given set, including the empty set and the set itself, express as  $P(A)$  or  $2^A$

- **Definition**  $P(A) = \{ x \mid x \subseteq A \}$

**Example:**

$$P(\emptyset) = \{\emptyset\},$$

$$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$$

$$P(\{1, \{2, 3\}\}) = \{\emptyset, \{1\}, \{\{2, 3\}\}, \{1, \{2, 3\}\}\}$$

- **Count**

**If  $|A| = n$ , then  $|P(A)| = 2^n$**

↳ Sets Operations •  $(\cup, \cap, -, \oplus, \sim)$ 

- Union:  $A \cup B = \{x \mid x \in A \vee x \in B\}$
- Intersection:  $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- Relative Complement:  $A - B = \{x \mid x \in A \wedge x \notin B\}$
- Symmetric Difference:  

$$A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$
- Absolute Complement:  $\sim A = E - A = \{x \mid x \notin A\}$

Design  $E = \{0, 1, \dots, 9\}$ ,  $A = \{0, 1, 2, 3\}$ ,  $B = \{1, 3, 5, 7, 9\}$ ,

Then:  $A \cup B = \{0, 1, 2, 3, 5, 7, 9\}$ ,  $A \cap B = \{1, 3\}$ ,  $A - B = \{0, 2\}$ ,

$A \oplus B = \{0, 2, 5, 7, 9\}$ ,  $\sim A = \{4, 5, 6, 7, 8, 9\}$ ,  $\sim B = \{0, 2, 4, 6, 8\}$

## ⓘ Instructions:

- (1) Use **only parentheses**.
- (2) **Order of Operations:**
  - Priority (1): Parentheses
  - Priority (2): Complement  $\sim$  and power set
  - Priority (3): Other operations.
  - Operations of the same priority are evaluated from left to right.

e.g. >>> **Example 1:** Let  $E = \{x \mid x \text{ is a student at a university in Beijing}\}$ ,  $A, B, C, D$  are subsets from  $E$ ,

$A = \{x \mid x \text{ is from Beijing}\}$ ,  $B = \{x \mid x \text{ is a external student}\}$ ,

$C = \{x \mid x \text{ is a student in the Department of Mathematics}\}$

$D = \{x \mid x \text{ likes listening to music}\}$ .

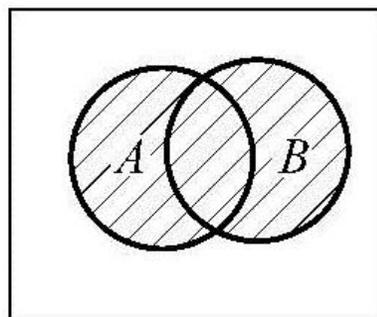
➤ **Describe the characteristics of the following sets of students:**

$(A \cup D) \cap \sim C = \{x \mid x \text{ is from Beijing or likes listening to music, but is not a mathematics student}\}$

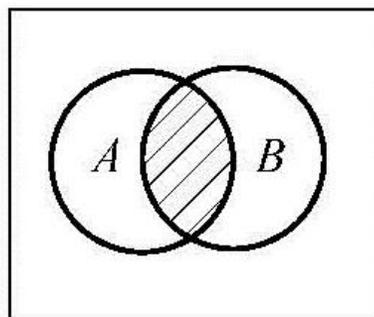
$\sim A \cap B = \{x \mid x \text{ is a non-local commuting student}\}$

$(A - B) \cap D = \{x \mid x \text{ is a boarding student in Beijing and likes listening to music}\}$

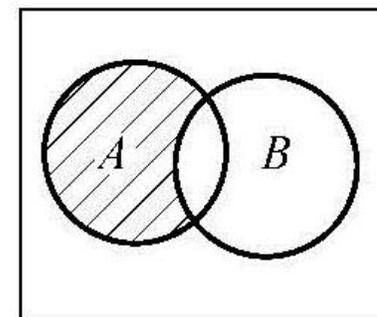
$\sim D \cap \sim B = \{x \mid x \text{ is a boarding student who does not like listening to music}\}$



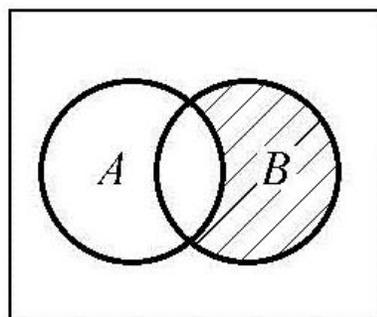
$A \cup B$



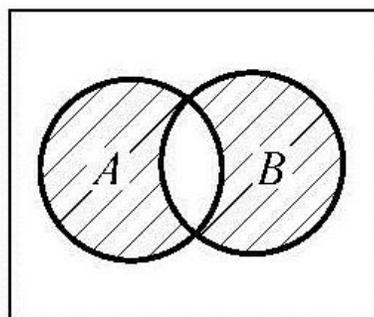
$A \cap B$



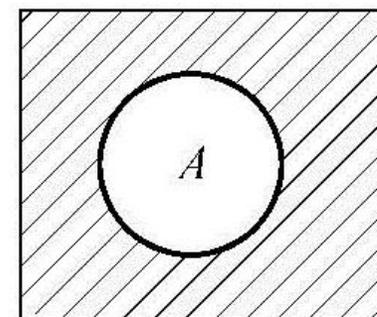
$A - B$



$B - A$



$A \oplus B$



$\sim A$

- Union and intersection operations can be extended to a **finite number** of sets

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \{x | x \in A_1 \vee x \in A_2 \vee \cdots \vee x \in A_n\}$$

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \cdots \cap A_n = \{x | x \in A_1 \wedge x \in A_2 \wedge \cdots \wedge x \in A_n\}$$

- Union and intersection operations can also be extended to countably **infinite** sets.

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots = \{x | \exists i (i = 1, 2, \dots) x \in A_i\}$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \cdots = \{x | \forall i (i = 1, 2, \dots) x \in A_i\}$$

*e.g.* >>> Example 2: Let  $A_i = [0, 1/i)$ ,  $B_i = (0, i)$ ,  $i=1, 2, \dots$ , then

$$\bigcup_{i=1}^n A_i = [0, 1)$$

$$\bigcap_{i=1}^n A_i = [0, 1/n)$$

$$\bigcup_{i=1}^n B_i = (0, n)$$

$$\bigcap_{i=1}^n B_i = (0, 1)$$

$$\bigcup_{i=1}^{\infty} A_i = [0, 1)$$

$$\bigcap_{i=1}^{\infty} A_i = \{0\}$$

$$\bigcup_{i=1}^{\infty} B_i = (0, +\infty)$$

$$\bigcap_{i=1}^{\infty} B_i = (0, 1)$$

1. Idempotent Law:  $A \cup A=A, A \cap A=A$
2. Associative Law:  $(A \cup B) \cup C=A \cup (B \cup C)$   
 $(A \cap B) \cap C=A \cap (B \cap C)$
3. Commutative Law:  $A \cup B=B \cup A, A \cap B=B \cap A$
4. Distributive Law:  $A \cup (B \cap C)=(A \cup B) \cap (A \cup C)$   
 $A \cap (B \cup C)=(A \cap B) \cup (A \cap C)$
5. Identity Law :  $A \cup \emptyset=A, A \cap E=A$
6. Zero Law :  $A \cup E=E, A \cap \emptyset=\emptyset, E$  is universal set
7. Law of Excluded Middle:  $A \cup \sim A=E$

8. Law of Contradiction:  $A \cap \sim A = \emptyset$

9. Absorption Law:  $A \cup (A \cap B) = A$ ,  $A \cap (A \cup B) = A$

10. De Morgan's Laws:

Absolute Form:  $\sim(B \cup C) = \sim B \cap \sim C$ ,  $\sim(B \cap C) = \sim B \cup \sim C$

Relative Form:  $A - (B \cup C) = (A - B) \cap (A - C)$

$A - (B \cap C) = (A - B) \cup (A - C)$

11. Complementation Law:  $\sim \emptyset = E$ ,  $\sim E = \emptyset$

12. Double Negation Law:  $\sim \sim A = A$

13. Complement and Intersection Conversion Law:  $A - B = A \cap \sim B$

## 14. Identities of Symmetric Difference

(1) Commutative Law:  $A \oplus B = B \oplus A$

(2) Associative Law :  $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

(3) Distributive Law of Intersection over Symmetric Difference:

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$

(4)  $A \oplus \emptyset = A, A \oplus E = \sim A$

(5)  $A \oplus A = \emptyset, A \oplus \sim A = E$

**Note:** The union operation  $\cup$  does not distribute over the symmetric difference  $\oplus$ , A counter example is given below:

$$A = \{a, b, c\}, B = \{b, c, d\}, C = \{c, d, e\}$$

$$A \cup (B \oplus C) = \{a, b, c\} \cup \{b, e\} = \{a, b, c, e\}$$

$$(A \cup B) \oplus (A \cup C) = \{a, b, c, d\} \oplus \{a, b, c, d, e\} = \{e\}, \text{ not equal}$$

15.  $\underline{A} \subseteq A \cup B, \quad \underline{B} \subseteq A \cup B.$

16.  $A \cap \underline{B} \subseteq A, \quad A \cap \underline{B} \subseteq B.$

17.  $\underline{A-B} \subseteq A.$

18.  $A \cup B = B \Leftrightarrow \underline{A} \subseteq B \Leftrightarrow A \cap B = A \Leftrightarrow A - B = \emptyset.$

19.  $A \oplus B = A \oplus C \Leftrightarrow B = C.$